

PROBLEM 4.20(a) $[L_z, x] = [xP_y - yP_x, x] = [xP_y, x] - [yP_x, x] = 0 - y[P_x, x] = i\hbar y.$ ✓

$[L_z, y] = [xP_y - yP_x, y] = [xP_y, y] - [yP_x, y] = x[P_y, y] - 0 = -i\hbar x.$ ✓

$[L_z, z] = [xP_y - yP_x, z] = [xP_y, z] - [yP_x, z] = 0 - 0 = 0.$ ✓

$[L_z, P_x] = [xP_y - yP_x, P_x] = [xP_y, P_x] - [yP_x, P_x] = P_y[x, P_x] = i\hbar P_y.$ ✓

$[L_z, P_y] = [xP_y - yP_x, P_y] = [xP_y, P_y] - [yP_x, P_y] = 0 - P_x[y, P_y] = -i\hbar P_x.$ ✓

$[L_z, P_z] = [xP_y - yP_x, P_z] = [xP_y, P_z] - [yP_x, P_z] = 0 - 0 = 0.$ ✓

(b) $[L_z, L_x] = [L_z, yP_z - zP_y] = [L_z, yP_z] - [L_z, zP_y] = [L_z, y]P_z - [L_z, z]P_y$
 $= -i\hbar xP_z + i\hbar zP_y = i\hbar(zP_x - xP_z) = i\hbar L_y.$ (So, by cyclic permutation of the indices, $[L_x, L_y] = i\hbar L_z).$

(c) $[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2] = [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y] + 0$
 $= i\hbar yx + xi\hbar y + (-i\hbar x)y + y(-i\hbar x) = \boxed{0}.$

$[L_z, p^2] = [L_z, P_x^2] + [L_z, P_y^2] + [L_z, P_z^2] = [L_z, P_x]P_x + P_x[L_z, P_x] + [L_z, P_y]P_y + P_y[L_z, P_y] + [L_z, P_z]P_z + P_z[L_z, P_z] + 0$
 $= i\hbar P_y P_x + P_x i\hbar P_y + (-i\hbar P_x)P_y + P_y (-i\hbar P_x) = \boxed{0}.$

(d) It follows from (c) that all three components of \vec{L} commute with r^2 and p^2 , and hence with $H = \frac{1}{2m}p^2 + V(\sqrt{r^2}).$ QED

PROBLEM 4.22 (a) $L + Y_l^l = 0$ (top of the ladder).

(b) $L_z Y_l^l = \hbar l Y_l^l \Rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \phi} Y_l^l = \hbar l Y_l^l,$ so $\frac{\partial Y_l^l}{\partial \phi} = i\hbar Y_l^l,$ and hence $Y_l^l = f(\theta) e^{i\hbar \phi}.$

[Note: $f(\theta)$ is the "constant" here - it's constant with respect to ϕ ... but still can depend on θ .]

$L_r Y_l^l = 0 \Rightarrow \hbar e^{i\hbar \phi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) [f(\theta) e^{i\hbar \phi}] = 0, \text{ or } \frac{df}{d\theta} e^{i\hbar \phi} + i \cot \theta i \hbar c^{i\hbar \phi} = 0, \text{ so}$

$\frac{df}{d\theta} = l \cot \theta f; \quad \frac{df}{f} = l \cot \theta d\theta \Rightarrow \int \frac{df}{f} = l \int \frac{\cos \theta}{\sin \theta} d\theta \Rightarrow \ln f = l \ln(\sin \theta) + \text{constant}.$

$\ln f = \ln(\sin l\theta) + K \Rightarrow \ln \left(\frac{f}{\sin l\theta} \right) = K \Rightarrow \frac{f}{\sin l\theta} = \text{constant} \Rightarrow f(\theta) = A \sin^{l\theta}.$

$\therefore Y_l^l(\theta, \phi) = A (e^{i\hbar \phi} \sin \theta)^l$

(c) $1 = A^l \int \sin^{2l} \theta \sin \theta d\theta d\phi = 2\pi A^l \int_0^\pi \sin^{(2l+1)} \theta d\theta = 2\pi A^l 2^l \frac{(2 \cdot 4 \cdot 6 \cdots (2l))}{1 \cdot 3 \cdot 5 \cdots (2l+1)}$

$= 4\pi A^l \frac{(2 \cdot 4 \cdot 6 \cdots (2l))^l}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2l+1)} = 4\pi A^l \frac{(2^l l!)^l}{(2l+1)!}. \quad \therefore A = \frac{1}{2^{l+1} l!} \sqrt{\frac{(2l+1)!}{\pi}}$

same as Problem 4.5,
except for $(-1)^l,$ which is
arbitrary anyway.

$$\begin{aligned}
 \text{Problem 4.23} \quad L Y_1^1 &= \hbar e^{i\phi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right) \left[-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \right] \\
 &= -\sqrt{\frac{15}{8\pi}} \hbar e^{i\phi} \left\{ e^{i\phi} (\cos^2\theta - \sin^2\theta) + i \frac{\cos\theta}{\sin\theta} \sin\theta \cos\theta i e^{i\phi} \right\} = -\sqrt{\frac{15}{8\pi}} \hbar e^{2i\phi} (\cos^2\theta - \sin^2\theta - \cos^2\theta) \\
 &= \sqrt{\frac{15}{8\pi}} \hbar (e^{i\phi} \sin\theta)^2 = \hbar \sqrt{2 \cdot 3 - 1 \cdot 2} Y_1^2 = 2\hbar Y_1^2. \quad \therefore Y_1^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} (e^{i\phi} \sin\theta)^2.
 \end{aligned}$$

$$\text{Problem 4.25 (a)} \quad H = 2\left(\frac{1}{2}mv^2\right) = mv^2; \quad |\vec{L}| = 2\frac{\pi}{2}mv = amv, \text{ so } L^2 = a^2 m^2 v^2. \quad \therefore H = \frac{L^2}{ma^2}.$$

But we know the eigenvalues of L^2 : $\hbar^2 l(l+1)$ — or, since we usually label energies with n :

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2} \quad (n=0,1,2,\dots)$$

(b) $\psi_{n,m}(\theta, \phi) = Y_n^m(\theta, \phi)$ — the ordinary spherical harmonics. The degeneracy of the n^{th} energy level is the number of m -values for given n : $2n+1$.

Problem 4.27

$$\begin{aligned}
 (a) [S_x, S_y] &= S_x S_y - S_y S_x = \frac{\hbar^2}{4} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = \frac{\hbar^2}{4} \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\} \\
 &= \frac{\hbar^2}{4} \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = i\hbar \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar S_z. \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \sigma_x \sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = \sigma_y \sigma_y = \sigma_z \sigma_z, \quad \text{so} \quad \sigma_j \sigma_j = 1 \quad \text{for } j=x, y, z. \\
 \sigma_x \sigma_y &= \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z; \quad \sigma_y \sigma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x; \quad \sigma_z \sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y; \quad \text{similarly} \quad \sigma_y \sigma_x = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z.
 \end{aligned}$$

$\sigma_z \sigma_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_x$; $\sigma_x \sigma_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y$. Equation [4.153] puts all this into a single equation. [Note: $\epsilon_{jkl} = \begin{cases} 1 & \text{if } jkl = xyz, yzx, \text{ or } zxy \\ -1 & \text{if } jkl = xzy, yxz, \text{ or } zyx \end{cases}$ and zero otherwise.]

PROBLEM 4.28 (a) $\chi^\dagger \chi = |A|^2 (9+16) = 25 |A|^2 = 1 \Rightarrow A = \frac{1}{5}$.

$$(b) \langle S_x \rangle = \chi^\dagger S_x \chi = \frac{1}{25} \frac{\hbar}{2} (-3i - 4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-3i - 4) \begin{pmatrix} 4 \\ 3i \end{pmatrix} = \frac{\hbar}{50} (-12i + 12i) = 0.$$

$$\langle S_y \rangle = \chi^\dagger S_y \chi = \frac{1}{25} \frac{\hbar}{2} (-3i - 4) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-3i - 4) \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12) = -\frac{24}{50} \hbar = -\frac{12}{25} \hbar.$$

$$\langle S_z \rangle = \chi^\dagger S_z \chi = \frac{1}{25} \frac{\hbar}{2} (-3i - 4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-3i - 4) \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{\hbar}{50} (9 - 16) = -\frac{7}{50} \hbar.$$

$$(c) \langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4} \text{ (always, for spin } \frac{1}{2}), \text{ so } \sigma_{S_x}^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0, \text{ so } \sigma_{S_x} = \frac{\hbar}{2}.$$

$$\sigma_{S_y}^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\hbar\right)^2 = \frac{\hbar^2}{2500} (625 - 576) = \frac{49}{2500} \hbar^2 \Rightarrow \sigma_{S_y} = \frac{7}{50} \hbar.$$

$$\sigma_{S_z}^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\hbar\right)^2 = \frac{\hbar^2}{2500} (625 - 49) = \frac{576}{2500} \hbar^2 \Rightarrow \sigma_{S_z} = \frac{12}{25} \hbar.$$

$$(d) \sigma_{S_x} \sigma_{S_y} = \frac{\hbar}{2} \cdot \frac{7}{50} \hbar \geq \frac{7}{2} \frac{\hbar}{2} |\langle S_x \rangle| = \frac{7}{2} \cdot \frac{7}{50} \hbar; \checkmark \text{ (right at the uncertainty limit).}$$

$$\sigma_{S_x} \sigma_{S_z} = \frac{7}{50} \hbar \cdot \frac{12}{25} \hbar \geq \frac{7}{2} \frac{\hbar}{2} |\langle S_z \rangle| = 0; \checkmark \text{ (trivial).}$$

$$\sigma_{S_y} \sigma_{S_z} = \frac{12}{25} \hbar \cdot \frac{7}{50} \hbar \geq \frac{7}{2} \frac{\hbar}{2} |\langle S_y \rangle| = \frac{7}{2} \cdot \frac{12}{25} \hbar; \checkmark \text{ (right at the uncertainty limit).}$$

$$\underline{\text{PROBLEM 4.29}} \quad \langle S_x \rangle = \frac{\hbar}{2} (a^* b^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} (a^* b^*) \begin{pmatrix} b \\ a \end{pmatrix} = \boxed{\frac{\hbar}{2} (a^* b + b^* a)} = \hbar \operatorname{Re}(ab^*).$$

$$\langle S_y \rangle = \frac{\hbar}{2} (a^* b^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} (a^* b^*) \begin{pmatrix} -ib \\ ia \end{pmatrix} = \frac{\hbar}{2} (-ia^* b + ib^* a) = \boxed{\frac{\hbar}{2} i(ab^* - a^* b)} = -\hbar \operatorname{Im}(ab^*).$$

$$\langle S_z \rangle = \frac{\hbar}{2} (a^* b^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} (a^* b^*) \begin{pmatrix} a \\ -b \end{pmatrix} = \frac{\hbar}{2} (a^* a - b^* b) = \boxed{\frac{\hbar}{2} (|a|^2 - |b|^2)}.$$

$$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4}; \quad S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}; \quad S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4}; \text{ so}$$

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}. \quad \langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{3}{4} \hbar^2 \stackrel{?}{=} s(s+1) \hbar^2 = \frac{1}{2} (\frac{1}{2} + 1) \hbar^2 = \frac{3}{4} \hbar^2 = \langle S^2 \rangle. \checkmark$$

$$\underline{\text{PROBLEM 4.30}} \quad (a) S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad \det \begin{pmatrix} -\lambda & -i\hbar \\ i\hbar & -\lambda \end{pmatrix} = \lambda^2 - \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm \frac{\hbar}{2}} \text{ (of course).}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow -i\beta = \pm \alpha; \quad |\alpha|^2 + |\beta|^2 = 1 \Rightarrow |\alpha|^2 + |\beta|^2 = 1 \Rightarrow \alpha = \pm \frac{1}{\sqrt{2}}.$$

$$\boxed{\chi_+^{(s)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \chi_-^{(s)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}.$$

$$(b) c_+ = \chi_+^{(s)\dagger} \chi = \frac{1}{\sqrt{2}} (1 - i) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} (a - ib). \quad \boxed{+ \frac{\hbar}{2}, \text{ with probability } \frac{1}{2} |a - ib|^2}.$$

$$c_- = \chi_-^{(s)\dagger} \chi = \frac{1}{\sqrt{2}} (1 + i) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} (a + ib). \quad \boxed{- \frac{\hbar}{2}, \text{ with probability } \frac{1}{2} |a + ib|^2}.$$

$$P_+ + P_- = \frac{1}{2} [(a^* + ib^*)(a - ib) + (a^* - ib^*)(a + ib)] = \frac{1}{2} [|a|^2 - ia^* b + iab^* + |b|^2 + |a|^2 + ia^* b - ib^* a + |b|^2] = |a|^2 + |b|^2 = 1. \checkmark \quad (c) \boxed{\frac{\hbar^2}{4}}, \text{ probability } \boxed{1}.$$